数理解析研究所講究録
22巻 1967年 10-17

Finite Type System of Partial Differential Operators and Decomposition of Solutions of Partial Differential Equations(*)

by

Shigetake MATSUURA

§1. Introduction, $\mathcal{P}(D)$-convexity.

First we fix some notaions. Let Ω be an open set in the n-dimensional Euclidean space $\mathbb{R}^n$ whose points shall be denoted by their coordinates $x = (x_1,\cdots,x_n)$. Let $\mathbb{C}[X_1,\cdots,X_n]$ be the polynomial ring in n variables $X = (X_1,\cdots,X_n)$ over the complex number field $\mathbb{C}$.

A partial differential operator with constant coefficients $P(D)$ is obtained from the polynomial $P(X) \in \mathbb{C}[X_1,\cdots,X_n]$ just by replacing the variables $X = (X_1,\cdots,X_n)$ by the differentiations $D = (D_1,\cdots,D_n)$ with $D_j = \frac{\partial}{\partial x_j}$ $(j = 1,2,\cdots,n)$.

Now let $\mathcal{P}(X) = (P_{jk}(X))$ be a matrix with q rows and p columns with coefficients in $\mathbb{C}[X_1,\cdots,X_n]$. $\mathcal{Q}(X) = (Q_{jk}(X))$ be a relation matrix with r rows and q columns for $\mathcal{P}(X)$, i.e. the row vectors $(Q_{j1}(X),\cdots,Q_{jq}(X))$ $(j = 1,\cdots,r)$ generat the relation module

(*) The content of this article was partly spoken in a slightly different form in Séminaire MALGRANGE (1964) à Orsay.

$$\{(Q_1(X),\cdots,Q_q(X)) \ ; \ \sum_{j=1}^{q} Q_j(X)P_{jk}(X) = 0 \quad (k = 1,\cdots,p) \quad .$$

Then we get the following differential complex

$$(1.1) \qquad [C^\infty(\Omega)]^p \xrightarrow{\mathcal{P}(D)} [C^\infty(\Omega)]^q \xrightarrow{\mathcal{Q}(D)} [C^\infty(\Omega)]^r \ ,$$

$$\mathcal{Q}(D)\,\mathcal{P}(D) = 0 \ .$$

DEFINITION. An open set Ω in $\mathbb{R}^n$ is called $\mathcal{P}(D)$-*convex* if the above sequence is exact.

EXAMPLES.

1) For the case of a single operator $P(D)$ $(p = q = 1)$, the exactness of (1.1) is reduced to the subjectivity

$$(1.2) \qquad P(D)C^\infty(\Omega) = C^\infty(\Omega) \quad ,$$

since the relation module is 0. Thus the concept of $\mathcal{P}(D)$-convexity is a generalization of that of the usual $P(D)$-convexity (see [3], [4]).

2) When $\mathcal{P}(D) = \begin{pmatrix} D_1 \\ \cdot \\ \cdot \\ \cdot \\ D_n \end{pmatrix}$ is the gradient operator, the exactness is just the vanishing of the first de Rham cohomology group

$$(1.3) \qquad H^1(\Omega, \mathbb{C}) = 0 \ .$$

3) If we identify the complex space $\mathbb{C}^m$ with $\mathbb{R}^n$ $(n = 2m)$ in the usual manner, then the pseudo-convexity of an open set Ω can be characterized by the exactness of the following sequence

$$(1.4)\quad C^{\infty}_{(0,0)}(\Omega) \xrightarrow{\bar\partial} C^{\infty}_{(0,1)}(\Omega) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} C^{\infty}_{(0,m)}(\Omega) \longrightarrow 0$$

where $C^{\infty}_{(0,q)}$ denotes the space of all C^{∞}-forms of type $(0,q)$ defined on Ω (see [2], [5]). This can be stated as the exactness of the following sequence

$$(1.4)' \qquad E_1 \xrightarrow{\bar\partial} E_2 \xrightarrow{\bar\partial} E_3$$

with $E_1 = \prod_{q=0}^{m-1} C^{\infty}_{(0,q)}(\Omega)$, $E_2 = \prod_{q=1}^{m} C^{\infty}_{(0,q)}(\Omega)$, $E_3 = \prod_{q=2}^{m+1} C^{\infty}_{(0,q)}(\Omega)$. Thus it can be written in the form (1.1).

4) When $\mathcal{P}(D)$ is quite general, we know only the following

THEOREM 1.1. (Ehrenpreis-Malgrange) *If Ω is convex, then Ω is $\mathcal{P}(D)$-convex for any $\mathcal{P}(D)$* (See [1], [2], [5]).

§2. Finite type systems of partial differential operators.

In the preceeding section, we have explained the $\mathcal{P}(D)$-convexity and some examples. But, other than these, no general results are known. Therefore it will have some meaning to state the following

THEOREM 2.1. *If Ω is simply connected, then Ω is $\mathcal{P}(D)$-convex for any $\mathcal{P}(D)$ of finite type.*(*)

Before giving a sketchy proof of the above theorem, we should explain some notions.

To a matrix $\mathcal{P}(X) = (P_{jk}(X))$ with q rows and p columns we associate the ideal $\mathscr{U} = \mathscr{U}(\mathcal{P})$ generated by all the (p,p)-minors of $\mathcal{P}(X)$. (If $p > q$, we only put $\mathscr{U} = 0$.) Let $V = V(\mathcal{P})$ be the algebraic variety defined by $\mathscr{U}(\mathcal{P})$, i.e.

$$V(\mathcal{P}) = \{\zeta \in \mathbb{C}^n \; ; \; P(\zeta) = 0 \quad \text{for all} \quad P \in \mathscr{U}(\mathcal{P})\} .$$

(*) See the definition below.

This is called the *variety attached* to the system of differential operators $\mathcal{P}(D)$.

DEFINITION. A system of partial differential operators $\mathcal{P}(D)$ is called *of finite type* if the attached variety is of dimension 0, i.e. $V(\mathcal{P})$ consists of only a finite number of points.

Now consider the following homogeneous equation

$$\mathcal{P}(D)U = 0 \tag{2.1}$$

where U is an unknown element of $[C^{\infty}(\Omega)]^p$

Then, using Hilbert's Nullstellensatz, it is not difficult to show the following

LEMMA *The vector space over* $\mathbb{C}$ *of the solutions of* (2.1) *is of finite dimension if and only if the system* $\mathcal{P}(D)$ *is of finite type. And then the solutions of* (2.1) *consist only of entire functions, more precisely the exponential-polynomial solutions of* (2.1).

To prove Theorem 2.1, first we fix a covering of Ω by its convex open subsets: $\Omega = \bigcup_i \Omega_i$, and consider the equation

$$\mathcal{P}(D)U = F \tag{2.2}$$

where F is an arbitrary given element in $[C^{\infty}(\Omega)]^q$ such that

$$\mathcal{Q}(D)F = 0 \ . \tag{2.3}$$

In each convex open set Ω_i, we can apply Theorem 1.1. and we get a solution in Ω_i. In the intersection $\Omega_i \cap \Omega_i$ of two such convex open sets, the difference of the solutions should satisfy the homogeneous equation (2.1). Hence, according to the above lemma, we get a solution of (2.2) in the union $\Omega_i \cup \Omega_j$ by adjusting those solutions by a certain entire function. Now starting at some fixed point we can proceed along curves repeating such adjustments in a manner similar to the usual analytic continuation in function theory. The resulting solution U should be univalent according to the assumption that Ω be simply connected. This shows that (1.1) is exact. This completes the proof.

§3. Decomposition of solutions of partial differential equations.

Since partial differential operators with constant coefficients operate on $C^\infty(\Omega)$ commutatively, we see clearly that

(I) For any pair of polynomials P_1, P_2 and for any pair of functions u_1, $u_2 \in C^\infty(\Omega)$ such that

$$P_1(D)u_1 = 0 , \quad P_2(D)u_2 = 0 , \tag{3.1}$$

the sum

$$u = u_1 + u_2 \tag{3.2}$$

satisfies the equation for the product operator

$$P_1(D)P_2(D)u = 0 ; \tag{3.3}$$

(II) For any polynomial P and for any multiindexed family of solutions $u_\alpha \in C^\infty(\Omega)$ $(|\alpha| \leq \nu-1)^{(*)}$ of the equation

(*) For a multiindex $\alpha = (\alpha_1, \cdots, \alpha_n)$ (a sequence of non-negative integers) we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $x = (x_1, \cdots, x_n)$ is variable point, we set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

(3.4) $$P(D)u_\alpha = 0$$

the sum

(3.5) $$u = \sum_{|\alpha|\leq\nu-1} x^\alpha u_\alpha$$

is a solution of the equation for the ν-times interacted operator

(3.6) $$P(D)^\nu u = 0 \quad .$$

Our question is to ask when the converses of the above facts are true, naturally assuming always that

(i) P_1 *and* P_2 *have no common factor*

for the converse of (I), and that

(ii) P *is irreducible*

for the converse of (II).

THEOREM 3.1. *If* Ω *is* $\mathcal{P}(D)$*-convex with* $\mathcal{P}(D) = \begin{pmatrix} P_1(D) \\ P_2(D) \end{pmatrix}$, *then any solution* $u \in C^\infty(\Omega)$ *of* (3.3) *can be decomposed into the form* (3.2) *with* (3.1). *Conversely, if each solution* u *of* (3.3) *can be written in the form* (3.2) *with* (3.1) *and if* Ω *is* $P_1(D)P_2(D)$*-convex, then* Ω *is* $\mathcal{P}(D)$*-convex.*

DEFINITION. We say that an open set Ω is a *decomposing* domain if the converses of (I) and (II) are true for all P_1, P_2 and P satisfying the conditions (i) and (ii).

Application of Ehrenpreis-Malgrange's theorem thus shows

THEOREM 3.2. *If* Ω *is convex, then* Ω *is a decomposing domain.*

In the plane $\mathbb{R}^2$, we can prove a more precise result, using Theorem 2.1.

THEOREM 3.3. *An open set* Ω *in* $\mathbb{R}^2$ *is a decomposing domain if and only if* Ω *is simply connected.*

The "only if" part comes from the fact that, for an open set Ω in the plane $\mathbb{R}^2$, the vanishing of the first de Rham cohomology group (1.3) is equivalent to the simple-connectedness of Ω.(*)

Theorem 3.3 is no longer true for $n \geq 3$.

If we restrict ourselves to the polynomial or exponential polynomial solutions then we get, by a purely algebraic argument, the following

THEOREM 3.4. *If we assume* (i) *and* (ii), *then any polynomial (resp. exponential-polynomial) solution of* (3.3) *or* (3.6) *can be decomposed into the form* (3.2) *or* (3.5) *with* u_1, u_2 *or* u_α *polynomials (resp. exponential-polynomials).*

The content of the present article can be considered as a completion of our previous work [6]. The more detailed treatment shall be published elsewhere.

References.

[1] Ehrenpreis, L., A fundamental principle for systems of linear differential equations with constant coefficients and some of its applications, Proc. Intern. Symp. on Linear Spaces, 161-174, Jerusalem, 1961.

[2] Hörmander, L., An introduction to complex analysis in several variables, D. Van Nostrand, Princeton, N. J., 1966.

(*) This can be proved by a combinatorial argument, for that I thank Prof. H. TODA.

[3] Hörmander, L., Linear partial differential oeprators, Springer-Verlag, 1963.

[4] Malgrange, B., Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution, Ann. Inst. Fourier 6, 271-354.(1955).

[5] Malgrange, B., Sur les systèmes différentiels à coefficients constants, Coll. C. N. R. S., 113-122, Paris, 1963.

[6] Matsuura, S., Factorization of differential operators and decomposition of solutions of homogeneous equations, Osaka Math. J., 213-231, 15 (1963).